Two-population replicator dynamics and number of Nash equilibria in random matrix games

Tobias Galla†‡

† The Abdus Salam International Center for Theoretical Physics, Strada Costiera 11, 34014 Trieste, Italy

‡INFM-CNR, Trieste-SISSA Unit, V. Beirut 2-4, 34014 Trieste, Italy

Abstract. We study the connection between the evolutionary replicator dynamics and the number of Nash equilibria in large random bi-matrix games. Using techniques of disordered systems theory we compute the statistical properties of both, the fixed points of the dynamics and the Nash equilibria. Except for the special case of zero-sum games one finds a transition as a function of the so-called co-operation pressure between a phase in which there is a unique stable fixed point of the dynamics coinciding with a unique Nash equilibrium, and an unstable phase in which there are exponentially many Nash equilibria with statistical properties different from the stationary state of the replicator equations. Our analytical results are confirmed by numerical simulations of the replicator dynamics, and by explicit enumeration of Nash equilibria.

1. Introduction

Replicator equations (RE) describe the evolution of populations of species interacting through co-operation and competition. A fitness is assigned to each species, dependent on the composition the population and on the availability of resources, and species fitter than average increase in concentration while the weight of species less fit than average decreases in time. Replicator dynamics (RD) have found widespread applications in game theory and economics as well as in population dynamics where an equivalence to Lotka-Volterra equations of theoretical biology can be established [1].

In the context of evolutionary game theory [2] RE describe games which are played repeatedly and where at any time-step the interaction is between individuals randomly chosen out of populations of agents. Each player is here taken to follow a pre-programmed strategy which they cannot change. Over time a payoff is accrued, and agents reproduce as described above, passing on their strategy to their descendents. This mechanism can be seen as reinforcement learning from past experience [3].

Prior to the launch of this evolutionary approach to game theory the analysis of games was mostly concerned with static aspects [4], i.e. with strategically optimal actions by

fully rational players and the characterization of potential equilibrium points. It is here assumed that the game is played only once. Players choose potentially stochastically from a set of available actions, and payoffs are then paid to each participant dependent on the decisions of all involved players. The key notion of a Nash equilibrium (NE) then refers to a point in strategy space so that no player can increase his payoff by unilaterally deviating from this point.

In this paper we extend existing work on single-population random replicator dynamics [5] in which a transition between stable and unstable regimes has been found as a function of a so-called 'co-operation pressure', and study the case of multi-population models, i.e. interaction between *distinct* populations of species. These correspond to so-called 'asymmetric' games [1] in which there is more than one type of player, such as in the game known as 'the battle of the sexes' in which male and female types of agents have different strategy sets at their disposal. In particular we study the stability of two-population replicator systems, and the relation to the number of NE in the corresponding bi-matrix games. We here extend the work of Berg et al [6], who computed the number and statistics of NE of matrix games with random payoff matrices in the absence of co-operation pressure and who identified an exponentially large number of NE for all types of payoff correlations considered, except for the case of zero-sum games in which there is a unique NE. Our work thus extends the statistical mechanics approach to replicator systems to the case of multiple populations, and establishes a connection between the earlier works of [5] and [6].

2. Model definitions

We will consider bi-matrix games between two players, one of type X and one of type Y, with strategy sets Σ_x and Σ_y, and payoff functions $\mu_x, \mu_y : \Sigma_x \times \Sigma_y \to \mathbb{R}$. We will here restrict to the case in which $|\Sigma_x| = |\Sigma_y| = N$. Strategies available to players of type X are labelled by $i \in \{1, \ldots, N\}$, the ones available to Y by $j \in \{1, \ldots, N\}$. The extension to more general cases with different numbers of available strategies is straightforward. Mixed strategies are then probability distributions over Σ_x and Σ_y respectively, described by vectors $\mathbf{x} = (x_1, \ldots, x_N)$ and $\mathbf{y} = (y_1, \ldots, y_N)$, with $0 \leq x_i \leq 1$ and $\sum_i x_i = 1$, and similarly for the $\{y_j\}$. Pure strategies are recovered as unit vectors. If player Y plays mixed strategy $\mathbf{y}$ then the expected payoff for X's pure strategy i reads $\nu_i^x(\mathbf{y}) = \sum_j \mu(i, j) y_j$. If both players play mixed strategies the expected payoff for X is denoted by $\nu_x(\mathbf{x}, \mathbf{y}) = \sum_i x_i \nu_i^x(\mathbf{y})$ and similarly for $\nu_y(\mathbf{x}, \mathbf{y})$. A NE is then a point $(\mathbf{x}^\star, \mathbf{y}^\star)$ so that $\nu_x(\mathbf{x}^\star, \mathbf{y}^\star) = \max_{\mathbf{x}} \nu_x(\mathbf{x}, \mathbf{y}^\star)$ and $\nu_y(\mathbf{x}^\star, \mathbf{y}^\star) = \max_{\mathbf{y}} \nu_x(\mathbf{x}^\star, \mathbf{y})$, and may be characterised by the following conditions

$$x_i^\star \left[\nu_i^x(\mathbf{y}^\star) - \nu_x(\mathbf{x}^\star, \mathbf{y}^\star) \right] = 0, \quad \nu_i^x(\mathbf{y}^\star) - \nu_x(\mathbf{x}^\star, \mathbf{y}^\star) \leq 0, \quad \forall i \qquad (1)$$

$$y_j^\star \left[\nu_j^y(\mathbf{x}^\star) - \nu_y(\mathbf{x}^\star, \mathbf{y}^\star) \right] = 0, \quad \nu_j^y(\mathbf{x}^\star) - \nu_y(\mathbf{x}^\star, \mathbf{y}^\star) \leq 0, \quad \forall j. \qquad (2)$$

The inequalities here ensure that no pure strategy delivers a higher payoff than obtainable at the Nash point, and the equalities enforce that all pure strategies which

are actually played with non-zero probability yield the same payoff, and that all other pure strategies are not employed.

RE of evolutionary game theory assume large populations of X and Y-type players, respectively, with each individual playing a pre-specified pure strategy. The replicators are hence the pure strategies of the game under consideration and are copied without error from parent to child. $x_i(t)$ denotes the relative concentration of pure strategy i in the X-population, and similarly for $y_j(t)$. The replicator equations describing the evolution of the X and Y populations are then as follows

$$\dot{x}_i(t) = x_i(t)\left[\nu_i^x(\mathbf{y}(t)) - \kappa_x(t)\right], \qquad \dot{y}_i(t) = y_i(t)\left[\nu_j^y(\mathbf{x}(t)) - \kappa_y(t)\right] \tag{3}$$

where the $x_i(t)$ and $y_j(t)$ are now time-dependent variables, and where $\kappa_x(t) = \sum_i x_i(t)\nu_i^x(\mathbf{y}(t))$, and similarly for $\kappa_y(t)$ are the average payoffs of the X and Y type players respectively. The replicator equations preserve the overall normalisation $\sum_i x_i(t) = \sum_j y_j(t) = 1$ in time. In terms of population dynamics the weights $\{x_i(t)\}$ and $\{y_j(t)\}$ correspond to relative concentrations of species, and the payoff functions ν_i^x an ν_j^y may be seen as their respective fitnesses. In a biological setting RD thus describe the temporal evolution of populations of species, where the concentrations of species fitter than average grow in and where all other species decrease in relative numbers. In the remainder of the paper we will use both the game theoretical and the population dynamical language synonymously. Note that in the two-population system species of type X interact only with species of type Y and vice versa.

We will in the following be concerned with replicator systems where all entries of the payoff matrices are Gaussian random variables. Following Peschel and Mende [1] as well as [5] we will also introduce so-called 'co-operation pressures' $u_x \geq 0$ and $u_y \geq 0$, and will consider payoff functions of the form

$$\nu_i^x(\mathbf{x}, \mathbf{y}) = -2u_x x_i + \sum_j a_{ij} y_j, \quad \nu_j^y(\mathbf{x}, \mathbf{y}) = -2u_y y_j + \sum_i b_{ji} x_i. \tag{4}$$

The role of u_x, u_y will be clarified below. For $u_x = u_y = 0$ one recovers the cases studied in [6] (with payoff matrices $a_{ij} = \mu_x(i,j)$, $b_{ji} = \mu_y(i,j)$). Both the static and dynamical properties of bi-matrix games are invariant under global rescaling and shifts of all payoff matrix elements. Without loss of generality we may therefore assume that the Gaussian variables a_{ij} and b_{ji} are drawn from the following statistics

$$\overline{a_{ij}^2} = \overline{b_{ji}^2} = \frac{1}{N}, \qquad \overline{a_{ij}b_{kl}} = \delta_{il}\delta_{jk}\frac{\Gamma}{N} \tag{5}$$

(with $\overline{\cdots}$ an average over the distribution of payoffs). The scaling with N is here introduced to guarantee a well defined thermodynamic limit. The parameter Γ characterises the correlations between the payoff matrices for the two different types of players. If $\Gamma = -1$ then $a_{ij} = -b_{ji}$ so that the resulting game is a zero-sum game (at vanishing co-operation pressures), corresponding to a prey-predator relation in the population dynamical setting. For $\Gamma = 0$ one has uncorrelated payoff matrices, and in the fully symmetric case $\Gamma = 1$ the two interacting players always receive equal payoff. Note also that for convenience we will re-scale the x_i and y_j such that they obey the

normalisations $\sum_{i=1}^{N} x_i(t) = \sum_{j=1}^{N} y_j(t) = N$ at all times t. The co-operation pressures u_x and u_y finally control the growth of individual species, large values of u_x drive the configuration $\mathbf{x}$ into the the interior of the simplex defined by $\sum_i x_i = N, 0 \leq x_i \leq N$, and similarly for u_y, see the book by Peschel and Mende for further details [1]. Indeed, inspection of Eqs. (3) for $u_x, u_y \to \infty$ shows that the RD leads to the fixed point $x_i = y_j \equiv 1$ for all species when started from non-zero concentrations, corresponding to full co-operation and maximal diversity. In a game theoretic setting large values of u_x, u_y favour the use of mixed strategies as opposed to pure strategies located at the corners of the above simplices.

3. Dynamics: generating functional analysis

On here directly addresses the dynamics described by the replicator equations, and formulates an effective theory for macroscopic dynamical order parameters. The starting point of the analysis is the moment generating functional, defined as

$$Z[\boldsymbol{\psi}^x, \boldsymbol{\psi}^y] = \left\langle\!\!\left\langle \exp\left(i \int dt \Big\{ \sum_{i=1}^{N} \psi_i^x(t) x_i(t) + \sum_{j=1}^{N} \psi_j^y(t) y_j(t) \Big\} \right) \right\rangle\!\!\right\rangle, \quad (6)$$

where $\langle\!\langle \ldots \rangle\!\rangle$ denotes an average over trajectories of the system, i.e. over solutions of the replicator equations (3). As usual for disordered systems a closed, but implicit set of equations describing the temporal evolution of a small number of disorder-averaged macroscopic order parameters can be derived. In the thermodynamic limit these observables turn out to be given by the correlation functions $C_x(t,t'), C_y(t,t')$, the response functions $G_x(t,t'), G_y(t,t')$ and the Lagrange parameters $\kappa_x(t), \kappa_y(t)$. The correlation and response matrices for the X-population are given by

$$C_x(t,t') = \lim_{N\to\infty} N^{-1} \sum_{i=1}^{N} \overline{\langle\!\langle x_i(t) x_i(t') \rangle\!\rangle}, \quad G_x(t,t') = \lim_{N\to\infty} N^{-1} \sum_{i=1}^{N} \overline{\left\langle\!\!\left\langle \frac{\delta x_i(t)}{\delta \kappa_x(t')} \right\rangle\!\!\right\rangle}, (7)$$

and analogous definitions for C_y, G_y apply. These order parameters are to be determined self-consistently as averages $C_x(t,t') = \langle x(t) x(t') \rangle_\star$, $G_x(t,t') = \langle \delta x(t) / \delta \kappa_x(t) \rangle_\star$ (and analogously for C_y, G_y) over realisations of the following pair of coupled stochastic effective processes

$$\dot{x}(t) = -x(t) \left[2u_x x(t) - \Gamma \int_{t_0}^{t} dt' \, G_y(t,t') x(t') - \kappa_x(t) + \eta_x(t) \right] \quad (8)$$

$$\dot{y}(t) = -y(t) \left[2u_y y(t) - \Gamma \int_{t_0}^{t} dt' \, G_x(t,t') y(t') - \kappa_y(t) + \eta_y(t) \right] \quad (9)$$

with t_0 the starting point of the dynamics. The notation $\langle \ldots \rangle_\star$ refers to an average over the effective process, i.e. over realisations of the noise variables $\{\eta_x(t)\}$ and $\{\eta_y(t)\}$. The covariances of these noise variables are given by $\langle \eta_x(t) \eta_x(t') \rangle_\star = C_y(t,t'), \langle \eta_y(t) \eta_y(t') \rangle_\star = C_x(t,t')$ with no correlations between η_x and η_y. Finally, the Lagrange multipliers $\{\kappa_x(t)\}$ and $\{\kappa_y(t)\}$ have to be chosen such that the constraints $\langle x(t) \rangle_\star = 1$ and $\langle y(t) \rangle_\star = 1$ are fulfilled at any time $t \geq t_0$.

Further progress can be made by assuming a fixed point of the replicator equations, i.e. by inspecting time-independent solutions $x(t) \equiv x$, $y(t) \equiv y$ of the effective processes (leading to constant correlation functions $C_x(t, t') \equiv q_x$, $C_y(t, t') \equiv q_y$ and to stationary response functions $G_x(\tau), G_y(\tau)$). Similarly to [5] one derives the following equations characterising such fixed point states

$$\chi_x(2u_x - \Gamma\chi_y) = g_0(\Delta_x) \qquad \chi_y(2u_y - \Gamma\chi_x) = g_0(\Delta_y)$$
$$q_y^{-1/2}(2u_x - \Gamma\chi_y) = g_1(\Delta_x) \qquad q_x^{-1/2}(2u_y - \Gamma\chi_x) = g_1(\Delta_y)$$
$$(q_x/q_y)(2u_x - \Gamma\chi_y)^2 = g_2(\Delta_x) \qquad (q_y/q_x)(2u_y - \Gamma\chi_x)^2 = g_2(\Delta_y) \tag{10}$$

with $\Delta_x = \kappa_x/\sqrt{q_y}$, $\Delta_y = \kappa_y/\sqrt{q_x}$ and $g_n(\Delta) = \int_{-\infty}^{\Delta} dz \frac{e^{-z^2/2}}{\sqrt{2\pi}}(\Delta - z)^n$ for $n \in \{0, 1, 2\}$. χ_x is here the integrated response $\chi_x = \int dt G_x(\tau)$, and similarly for χ_y. A linear stability analysis shows that such fixed points become unstable at the point at which $(\chi_x\chi_y)^2 = \phi_x\phi_y$, leading to an unstable and non-ergodic phase at low co-operation pressures. For $u_x = u_y$, which we will mostly consider in the following, one finds $\chi_x = \chi_y$ and $q_x = q_y$, and Eqs. (10) as well as the stability condition reduce to those of a single-population replicator system studied in [5]. In the unstable phase solutions of (10) can no longer be expected to describe the stationary states of the RE accurately.

4. Statics: replica analysis

The starting point of the replica analysis of the statics of the model is the observation that for symmetric couplings, i.e. $a_{ij} = b_{ji}$ ($\Gamma = 1$) the replicator equations (3) can be written in the form $\dot{x}_i = x_i(\partial_{x_i}\mathcal{H}(\mathbf{x}, \mathbf{y}) - \kappa_x(t))$ and similarly for $\dot{y}_j$, with $\mathcal{H}(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\sum_{i,j=1}^{N}[x_iy_j(a_{ij} + b_{ji})] - u_x\sum_{i=1}^{N}x_i^2 - u_y\sum_{j=1}^{N}y_j^2$, so that the stationary states of the RD at $\Gamma = 1$ correspond to extrema of $\mathcal{H}$. The computation of these is straightforward and based on the evaluation of $-\beta f = \lim_{N\to\infty} N^{-1}\overline{\ln Z(\beta)}$. f is here the disorder-averaged free energy density at temperature $T = \beta^{-1}$. $Z(\beta)$ stands for the partition function corresponding to $\mathcal{H}$:

$$Z(\beta) = \left[\prod_{i=1}^{N}\int_0^{\infty} dx_i \prod_{j=1}^{N}\int_0^{\infty} dy_j\right] \delta\left(\sum_{j-1}^{N} x_j - N\right) \delta\left(\sum_{j-1}^{N} y_j - N\right) e^{-\beta\mathcal{H}(\mathbf{x}, \mathbf{y})}. \tag{11}$$

We will not report the details of the straightforward replica calculation, but will only note that a replica-symmetric ansatz leads to a set of equations identical to (10) at zero temperature and in the thermodynamic limit (with $\Gamma = 1$). For $\Gamma < 1$ no Lyapunov function $\mathcal{H}$ can be found and the replica approach is inapplicable.

5. Annealed calculation of the number of Nash equilibria

Finally, the number and statistics of the NE of the corresponding bi-matrix game may be computed by direct integration over phase space enforcing conditions (1,2) through suitable delta- and step-functions. It is here convenient to set $\widetilde{x}_i = x_i$ if $x_i > 0$ and

$\widetilde{x}_i = -2u_x x_i + \sum_j a_{ij} y_j - \kappa_x$ if $x_i = 0$, and similarly for $\widetilde{y}_j$ [6]. The above conditions (1) may then be written as

$$I_i^x(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) \equiv \widetilde{x}_i \Theta(-\widetilde{x}_i) - \left(-2u\widetilde{x}_i \Theta(\widetilde{x}_i) + \sum_j a_{ij}\widetilde{y}_j \Theta(\widetilde{y}_j) - \kappa_x \right) = 0 \qquad (12)$$

and (2) translates into $I_j^y(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) = 0$ with an analogous expression $I_j^y(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$. $\Theta(\cdot)$ is the step-function. The number of NE at payoffs κ_x and κ_y is then given by

$$\mathcal{N}(\kappa_x, \kappa_y) = \int D\widetilde{\mathbf{x}} D\widetilde{\mathbf{y}} \, \delta\left(\sum_i \widetilde{x}_i \Theta(\widetilde{x}_i) - N \right) \delta\left(\sum_j \widetilde{y}_i \Theta(\widetilde{y}_i) - N \right)$$

$$\times \prod_i \delta(I_i^x(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})) \prod_j \delta(I_j^y(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})) \, |\det D|. \qquad (13)$$

$\det D$ is a normalising determinant. Performing the disorder-average in an annealed approximation, one converts the computation into a saddle-point problem in the thermodynamic limit. We set $u_x = u_y = u$ for simplicity and find exponential domination at equal payoff $\kappa_x = \kappa_y = \kappa$ and $\lim_{N\to\infty} \frac{1}{N} \ln \overline{\mathcal{N}(\kappa)} = S(\kappa)$, where

$$S(\kappa) = \mathrm{extr}_{\{E,R,q,\widetilde{q},\phi,\widetilde{\phi}\}} \left\{ 2E - \Gamma R^2 + 2\widetilde{q}q + 2\widetilde{\phi}\phi + 2\ln\left[H\left(-\frac{\kappa}{\sqrt{q}} \right) + \frac{e^{-\widetilde{\phi}}}{\sqrt{(2u-\Gamma R)^2 + 2q\widetilde{q}}} \right. \right.$$

$$\left. \left. \times H\left(\frac{(2u-\Gamma R)\kappa/q + E}{\sqrt{q^{-1}(2u-\Gamma R)^2 + 2\widetilde{q}}} \right) \times \exp\left(-\frac{\kappa^2}{2q} + \frac{(E + (2u-\Gamma R)\kappa/q)^2}{4\widetilde{q} + 2(2u-\Gamma R)^2/q} \right) \right] + g \right\} \qquad (14)$$

We have here set abbreviated $H(x) = \frac{1}{2}\left(1 - \mathrm{erf}(x/\sqrt{2}) \right)$, and g denotes the contribution from the normalising determinant, which we do not report explicitly. For $u = 0$ we recover the result of [6]. Extremisation of (14) leads to an annealed upper bound of the logarithmic number of NE at payoff κ.

6. Results

Our results are summarised by figures 1 and 2. Fig. 1a shows the phase diagram of the bi-population replicator model at different symmetry parameters Γ. For $\Gamma > -1$ one finds a transition line, with unique fixed points of the RE at large co-operation pressures to the top-right, and an unstable regime below. For $\Gamma = -1$ no transition is observed, in this case there is one unique fixed-point of the dynamics for any (u_x, u_y). In the remaining figures we restrict to the case $u_x = u_y = u$. The transition between the stable and unstable phases then occurs at co-operation pressure $u_c(\Gamma) = (1+\Gamma)/(2\sqrt{2})$. In Fig. 1b we depict typical curves $S(\kappa)$ of the entropy of NE at payoff κ as obtained from Eq. (14). These curves typically show a maximum $S_m(u, \Gamma)$ at intermediate values of the payoff κ, indicating the number of dominating NE. If $S_m > 0$ a large number of Nash equilibria is present in the thermodynamic limit, while for $S_m < 0$ NE are exponentially suppressed. If $S_m = 0$ a single NE prevails [6]. The inset of Fig. 1b shows a comparison between the analytical results for $S(\kappa)$ at a fixed combination of u and Γ with numerical data from an explicit enumeration of NE. We here use a direct support
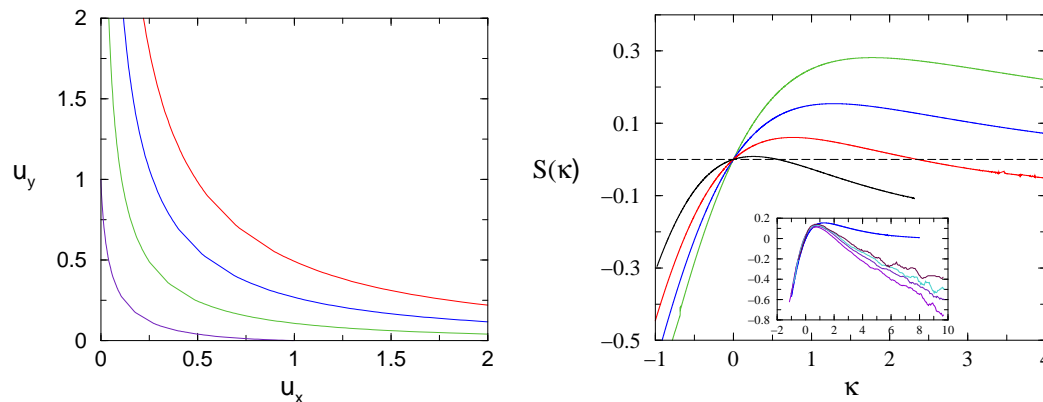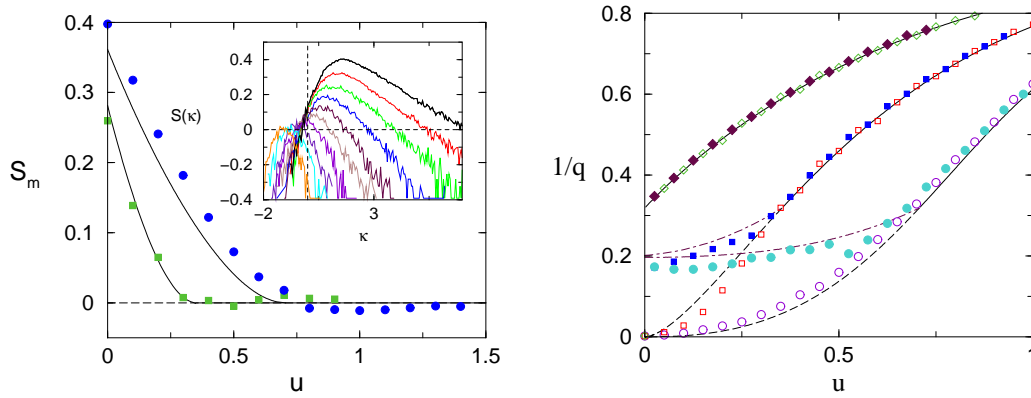
Figure 1.
Left: (Colour on-line) Phase diagram of the two-population replicator system. Lines separate the stable phase (up-right) from the instable one, and are plotted for $\Gamma = 1, 0.5, 0, -0.5$ from top to bottom.
Right: (Colour on-line) Main panel: entropy of number of Nash equilibria for the uncorrelated game $\Gamma = 0$ at $u = 0, 0.1, 0.2, 0.3$ (from top to bottom at the maximum). Inset: comparison with numerical data from support enumeration ($u = 0.1$). Upper curve is the theoretical line, others are from support enumeration of single-population models at $N = 14, 16, 18, 20$ (bottom to top). A running average has been performed to smoothen numerical data.

enumeration scheme, which we apply to an equivalent single-population game in order to reduce the required computing time. Accessible system sizes are limited to $N \approx 20$, and while results are consistent with the theory, finite size effects can be significant.

Fig. 2a shows the entropy S_m of the dominating NE as a function of u at fixed values of the symmetry parameter Γ. One finds a large number of NE ($S_m > 0$) for $u < u_c(\Gamma)$, and a unique NE above u_c ($S_m = 0$). At u_c the spectra $S(\kappa)$ take their maxima at zero payoff $\kappa = 0$. The transition point $u_c(\Gamma)$ coincides with the one separating the stable and unstable phases of the RD. Results from an explicit enumeration of NE are consistent with this transition although direct quantitative comparison between simulations and theory appears difficult due to finite size effects. An example of raw data obtained from enumeration of NE is shown in the inset of Fig. 2a.

The saddle point extremisation of (14) allows one also to compute the statistics of the dominating NE (in the annealed approximation) and to compare with the stationary states of the RE. We here focus on the order parameter q^{-1} which serves as a measure of the diversity of the eco-system [7], with large values of q^{-1} corresponding to many surviving species (equivalently many pure strategies played with non-zero probability). Measurements of q^{-1} in the stationary states of the RE and numerical results for the NE are shown in Fig. 2b. The latter are here obtained using a repeated Lemke-Howson algorithm [8]. Note that the fixed-point theory of the RE applies only above u_c but has been continued below as dashed lines. Quantitative deviations between numerical and theoretical results for NE are due to finite-size or sampling effects and the annealed approximation of the theory. The analytical and numerical results show

Figure 2.
Left: (Colour on-line) Entropy S_m of Nash equilibria as a function of u, for $\Gamma = 0$ (left curve) and $\Gamma = 1$ (right curve). Solid lines from theory, symbols from explicit enumeration of single-population systems with $N = 20$, averaged over 1000 samples. Inset: Raw data for $S(\kappa)$ at $\Gamma = 1$ for $u = 0.9, 0.8, \ldots, 0.0$ from left to right.
Right: (Colour on-line) Diversity parameter q^{-1} at $\Gamma = -1, 0, 1$ (from left to right). Solutions of (10) depicted as solid lines where replicator equations have stable fixed points, continued into the instable phase as dashed lines. Open symbols are results from simulations of bi-replicator systems with $2N = 1000$ species. Dashed-dotted curves show theory for Nash equilibria for $u \leq u_c(\Gamma)$, open symbols are numerical results for typical NE as obtained from iterated Lemke-Howson algorithm applied to bi-matrix systems of size $2N = 100$, averaged over 200 samples.

that the diversity of species in the NE coincides with that at the fixed points of the RD above $u_c(\Gamma)$, but that they differ from each other below the transition. Very similar results are found for the number of surviving species. We note that conditions (1,2) are necessary but not sufficient for the stability of fixed points of the RE so that stable fixed points are always NE but not necessarily vice versa. Results thus indicate that NE are statistically distinct from potential stable attractors of the dynamics below u_c (and more numerous). Note also that below u_c marginally stable fixed points are suppressed for $\Gamma < 1$ [5] and hence volatile, possibly chaotic behaviour is observed in simulations. For $\Gamma = 1$ the dynamics converges to a fixed point also below u_c, but is sensitive to initial conditions (see also [5, 9]).

7. Summary and concluding remarks

Our results may be summarised as follows: for games different from the zero-sum type (i.e. for any $\Gamma > -1$) one finds a dynamical instability of the fixed points of the two-population replicator equations at $u_c(\Gamma) = (1 + \Gamma)/(2\sqrt{2})$. For $\Gamma = 1$ this instability coincides with an AT-instability of the replica symmetric solution of the statics [10]. Above u_c all three approaches (dynamics, NE and replica theory where applicable) lead to the same order parameters, describing an ergodic state, in which there is one unique NE which coincides with the unique fixed point of the RD (and with the unique

extremum of $\mathcal{H}$ for $\Gamma = 1$). Below u_c ergodicity is broken, the stationary state of the RD is no longer unique, and there is an exponential number of NE. The statistics of the NE differs from those of the stationary states of the dynamics. The observations generalise the results of [6], which were concerned with the case of vanishing co-operation pressure u. At $u = 0$ the RD is instable for all $\Gamma > -1$, and marginally stable for zero-sum games, $\Gamma = -1$. Except for the latter case the system is therefore in the non-ergodic phase of the RE, in line with the reported exponential number of NE in [6]. At $\Gamma = -1$ one is precisely at the phase transition at $u_c(\Gamma = -1) = 0$ and finds a zero-entropy of dominating NE.

In conclusion we have investigated bi-matrix games and two-population replicator systems with tools of disordered systems theory, and have have computed the number and typical properties of NE and of the fixed points of the corresponding replicator equations. Several extensions of the present work can be considered, including cases of correlations between different rows and/or columns of the payoff matrices, populations with different numbers of strategies at hand and systems with more than two types of players, as well as the study of other stability concepts of evolutionary game theory in the context of random replicator systems.

Acknowledgements

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